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Renormalization in spherical field theory[1]

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We derive several results concerning non-perturbative renormalization in the spherical field formalism. Using a small set of local counterterms, we are able to remove all ultraviolet divergences in a manner such that the renormalized theory is finite and translationally invariant. As an explicit example we consider massless ϕ^4 theory in four dimensions. [PACS numbers: 11.10.-z, 11.10.Gh, 11.15.Tk]

1 Introduction

Spherical field theory is a non-perturbative method which uses the spherical partial wave expansion to reduce a general d-dimensional Euclidean field theory into a set of coupled radial systems ([1], [2]). High spin partial waves correspond with large tangential momenta and can be neglected if the theory is properly renormalized. The remaining system can then be converted into differential equations and solved using standard numerical methods. ϕ^4 theory in two dimensions was considered in [1]. In that case there was only one divergent diagram, and it could be completely removed by normal ordering. In general any super-renormalizable theory can be renormalized by removing the divergent parts of divergent diagrams. Using a high-spin cutoff $J_{\max}$ and discarding partial waves with spin greater than $J_{\max}$, we simply compute the relevant counterterms using spherical Feynman rules.

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The $J_{\max}$ cutoff scheme however is not translationally invariant. It preserves rotational invariance but regulates ultraviolet processes differently depending on radial distance. In the two-dimensional ϕ^4 example it was found that the mass counterterm had the form

$$\mathcal{L}_{c.t.} \propto \phi^2(\vec{t}) \left[K_0(\mu t)I_0(\mu t) + 2\textstyle\sum_{n=1,J_{\max}} K_n(\mu t)I_n(\mu t)\right], \tag{1}$$

where I_n, K_n are n^{th}-order modified Bessel functions of the first and second kinds, μ is the bare mass, and t is the magnitude of $\vec{t}$. As $J_{\max} \to \infty$, we find

$$\mathcal{L}_{c.t.} \propto \phi^2(\vec{t}) \left[\log(\tfrac{2J_{\max}}{\mu t}) + O(J_{\max}^{-1})\right]. \tag{2}$$

Our regularization scheme varies with t, and we see that the counterterm also depends on t. The physically relevant issue, however, is whether or not the renormalized theory is independent of t. In this case the answer is yes. Any t dependence in renormalized amplitudes is suppressed by powers of $J_{\max}^{-1}$, and translational invariance becomes exact as $J_{\max} \to \infty$.

We now consider general renormalizable theories, in particular those which are not super-renormalizable. In this case the number of divergent diagrams is infinite. Since we are primarily interested in non-perturbative phenomena, a diagram by diagram subtraction method is not useful. In the same manner strictly perturbative methods such as dimensional regularization are not relevant either. Our interest is in non-perturbative renormalization, where coefficients of renormalization counterterms are determined by non-perturbative computations.[4] In this paper we analyze the general theory of non-perturbative renormalization in the spherical field formalism. In the course of our analysis we answer the following three questions: (i) Can ultraviolet divergences be cancelled by a finite number of local counterterms? (ii) Can the renormalized theory be made translationally invariant? (iii) What is the general form of the counterterms?

The organization of the paper is as follows. We begin with a discussion of differential renormalization, a regularization-independent method which will allow us to construct local counterterms. Next we describe a regularization procedure which is convenient for spherical field theory. In the large radius limit $t \to \infty$ our regularization procedure (which we call angle smearing) is

[4] We should mention that Pauli-Villars regularization is compatible with non-perturbative renormalization. However this introduces additional unphysical degrees of freedom and tends to be computationally inefficient.

anisotropic but locally invariant under translations. For general t we expand in powers of t^{-1} to generate the general form of the counterterms. We conclude with two examples of one-loop divergent diagrams. We show by direct calculation that the predicted counterterms render these processes finite and translationally invariant.

2 Differential renormalization

Differential renormalization is the coordinate space version of the BPHZ method.[5] It is framed entirely in coordinate space, and renormalized amplitudes can be defined as distributions without reference to any specific regularization procedure. Differential renormalization was introduced in [3], and a systematic analysis of differential renormalization to all orders in perturbation theory using Bogoliubov's recursion formula was first described in [4]. The usual implementation of differential renormalization is carried out using singular Poisson equations and their explicit solutions. In our discussion, however, we find it more convenient to operate directly on the distributions.[6] We describe the details of our approach in the following. We should stress that the two approaches are equivalent, differing only at the level of formalism.

We assume that we are working with a renormalizable theory. For indices $i_1, \cdots i_j$ let us define

$$t^{i_1,\cdots i_j} = t^{i_1} t^{i_2} \cdots t^{i_j}, \tag{3}$$

$$\nabla_{i_1,\cdots i_j} = \nabla_{i_1} \nabla_{i_2} \cdots \nabla_{i_j}. \tag{4}$$

Let $f(\vec{t})$ be a smooth test function, and let $I(\vec{t}-\vec{t'};\mu^2)$ be a smooth function with support on a region of scale μ^{-1}. We define $S^j_{\vec{t'}}\left[f\right](\vec{t})$ as $I(\vec{t}-\vec{t'};\mu^2)$ multiplied by the j^{th} order term in the Taylor series of $f(\vec{t})$ about the point $\vec{t'}$. Inserting delta functions, we have

$$\begin{aligned} S^j_{\vec{t'}} f(\vec{t}) &= I(\vec{t}-\vec{t'};\mu^2) \sum_{i_1,\cdots i_j} \left[\tfrac{(t-t')^{i_1,\cdots i_j}}{j!} \nabla_{i_1,\cdots i_j} f(\vec{t'})\right] \\ &= I(\vec{t}-\vec{t'};\mu^2) \sum_{i_1,\cdots i_j} \tfrac{(t-t')^{i_1,\cdots i_j}}{j!} \textstyle\int d^4\vec{z}\; \nabla^{\vec{t'}}_{i_1,\cdots i_j} \delta^4(\vec{t'}-\vec{z})\, f(\vec{z}). \end{aligned} \tag{5}$$

[5] Paraphrase of private communication with Jose Latorre.

[6] Our approach is similar to the natural renormalization scheme described in [5]. In contrast with [5], however, we do not a priori specify the finite parts of amplitudes.

For the purposes of this discussion we will require

$$I(\vec{t} - \vec{t}'; \mu^2) = 1 + O^N(\vec{t} - \vec{t}') \quad \text{as } \vec{t}' \to \vec{t}, \tag{6}$$

where N is some positive integer greater than the superficial degree of divergence of any subdiagram[7] in the theory we are considering. For any renormalizable theory $N > 2$ will suffice. In our formalism, $I(\vec{t} - \vec{t}'; \mu^2)$ determines how finite parts of renormalized amplitudes are assigned, and μ is the renormalization mass scale.

We now consider a particular diagram, G, with n vertices. We define $K(\vec{t}_1, \cdots \vec{t}_n)$ to be the kernel of the amputated diagram, i.e., the value of the diagram with vertices fixed at points $\vec{t}_1, \cdots \vec{t}_n$. The amplitude is obtained by integrating $K(\vec{t}_1, \cdots \vec{t}_n)$ with respect to all internal vertices. We will regard K as a distribution acting on n smooth test functions $f_1, \cdots f_n$. (For external vertices containing more than one external line and/or derivatives, $f_{ext}(\vec{t}_{ext})$ should be regarded as a product of test functions, with possible derivatives, at $\vec{t}_{ext}$.)

$$K : f_1, \cdots f_n \to \int d^4\vec{t}_1 \cdots d^4\vec{t}_n \, K(\vec{t}_1, \cdots \vec{t}_n) f_1(\vec{t}_1) \cdots f_n(\vec{t}_n). \tag{7}$$

Let us assume that our diagram is primitively divergent with superficial degree of divergence j. We now define another distribution $T_G K$, which extracts the divergent part of K. We start with the case when G has more than one vertex. Let us define $T_G K : f_1, \cdots f_n \to$

$$\sum_{j_1 + \cdots + j_n \leq j} \int d^4\vec{t}_1 \cdots d^4\vec{t}_n \, K(\vec{t}_1, \cdots \vec{t}_n) S^{j_1}_{\vec{t}_{ave}} f_1(\vec{t}_1) \cdots S^{j_n}_{\vec{t}_{ave}} f_n(\vec{t}_n), \tag{8}$$

where $\vec{t}_{ave} = \frac{1}{n}(\vec{t}_1 + \cdots + \vec{t}_n)$. We note that the subtracted distribution $K - T_G K$ is finite and well-defined for all $f_1, \cdots f_n$. Let us define

$$F_K^{i_{1,1}, i_{2,1} \cdots i_{j_n,n}}(\vec{t}) \tag{9}$$
$$= \int d^4\vec{t}_1 \cdots d^4\vec{t}_n \, \delta^4\left(\frac{\vec{t}_1 + \cdots + \vec{t}_n}{n} - \vec{t}\right) K(\vec{t}_1, \cdots \vec{t}_n) \left[\prod_{k=1,\cdots n} \frac{I(\vec{t}_k - \vec{t}; \mu^2)(t_k - t)^{i_{1,k}, \cdots i_{j_k,k}}}{j_k!} \right].$$

[7] In our discussion a subdiagram is a subset of vertices together with all lines contained in those vertices.

We can then rewrite $T_G K : f_1, \cdots f_n \to$

$$\sum_{j_1+\cdots+j_n\le j} \sum_{\substack{i_{1,1},i_{2,1}\cdots \\ i_{1,n}\cdots i_{j_n,n}}} \left[\begin{array}{c} \int d^4\vec{t} F_K^{i_{1,1},i_{2,1}\cdots i_{j_n,n}}(\vec{t}) \int d^4\vec{z}_1 \cdots d^4\vec{z}_n \\ \left(\prod_{k=1,\cdots n} \nabla^{\vec{t}}_{i_{1,k},\cdots i_{j_k,k}} \delta^4(\vec{t}-\vec{z}_k)\right) f_1(\vec{z}_1)\cdots f_n(\vec{z}_n) \end{array} \right]. \tag{10}$$

The delta functions make this kernel completely local. We can read off the corresponding counterterm interaction by functional differentiation with respect to each of the component functions of $f_{ext}(\vec{t}_{ext})$ for the external vertices and setting $f_{int}(\vec{t}_{int}) = 1$ for the internal vertices. We now turn to the case when G has only one vertex. For this case we set $T_G K = K$, which is equivalent to normal ordering the interactions in our theory. In this case K is itself local and therefore $T_G K$ and our counterterm interaction are again local.

We now extend the definition of T_G in (10) to include the case of subdiagrams. Let G be a general 1PI diagram, and let G' be a renormalization part[8] of G with superficial degree of divergence j'. For notational convenience we will label the vertices of G so that the first n' vertices lie in G'. If G' has only one vertex then again we normal order the interaction. Otherwise we define $T_{G'} K : f_1, \cdots f_n \to$

$$\sum_{j'_1+\cdots+j'_{n'}\le j'} \int d^4\vec{t}_1 \cdots d^4\vec{t}_n \, K(\vec{t}_1, \cdots \vec{t}_n) \left[\begin{array}{c} S^{j'_1}_{\vec{t}_{ave}} f_1(\vec{t}_1) \cdots S^{j'_n}_{\vec{t}_{ave}} f_{n'}(\vec{t}_{n'}) \\ \cdot f_{n'+1}(\vec{t}_{n'+1}) \cdots f_n(\vec{t}_n) \end{array} \right], \tag{11}$$

where $\vec{t}_{ave} = \frac{1}{n'}(\vec{t}_1 + \cdots + \vec{t}_{n'})$.[9] This definition can be used recursively to define products of $T_{G'_1} T_{G'_2}$ for disjoint subdiagrams $G'_1 \cap G'_2 = \emptyset$ or nested subdiagrams $G'_1 \supset G'_2$. For the case of nested subdiagrams we always order the product so that larger diagrams are on the left.

It is straightforward to show that the T operation acts as the identity on local interactions and thus treats overlapping divergences in the same manner as BPHZ. Following the standard BPHZ procedure ([6]$-$[8]), we can write Bogoliubov's $\bar{R}$ operation using Zimmerman's forest formula,

$$\bar{R} = \sum_F \prod_{\gamma\in F} (-T_\gamma), \tag{12}$$

[8] A renormalization part is a 1PI subdiagram with degree of divergence ≥ 0.

[9] After applying $T_{G'}$, we regard G' as being contracted to single vertex at $\vec{t}_{ave}$.

where F ranges over all forests[10] of G, and γ ranges over all renormalization parts of F. In the product we have again ordered nested subdiagrams so that larger diagrams are on the left. Let j be the superficial degree of divergence of G. The renormalized kernel, RK, is given by

$$\begin{array}{rl} RK = \bar{R}K & \text{for } j < 0 \\ RK = (1 - T_G)\bar{R}K & \text{for } j \geq 0. \end{array} \tag{13}$$

Our final result is that all required counterterms are local, and the form of the counterterms is

$$\mathcal{L}_{c.t.} = \sum_{A(\phi,\nabla_i\phi)} F_A(\vec{t}) A(\phi(\vec{t}), \nabla_i\phi(\vec{t})), \tag{14}$$

where the sum is over operators of renormalizable type. For the case of gauge theories, our renormalization procedure is supplemented by the additional requirement that the renormalized amplitudes satisfy Ward identities.[11] If our regularization procedure breaks gauge invariance these identities are not automatic and the required local counterterms will in general be any operators of renormalizable type (not merely gauge-invariant operators). This is, however, a separate discussion, and the details of implementing Ward identity constraints will be discussed in future work.

3 Regularization by angle smearing

In this section we determine the functional form of the coefficients $F_A(\vec{t})$ in (14). To make the discussion concrete, we will illustrate using the example of massless ϕ^4 theory in four dimensions

$$\mathcal{L} = \tfrac{1}{2}\phi\nabla^2\phi - \tfrac{\lambda}{4!}\phi^4 + \mathcal{L}_{c.t.}. \tag{15}$$

From (14) $\mathcal{L}_{c.t.}$ is given by

$$F_{\phi^2}(\vec{t})\phi^2(\vec{t}) + \sum_{i,j} F^{ij}_{\nabla\phi\nabla\phi}(\vec{t})\nabla_i\phi(\vec{t})\nabla_j\phi(\vec{t}) + F_{\phi^4}(\vec{t})\phi^4(\vec{t}). \tag{16}$$

Let $G(\vec{t}, \vec{t}')$ be the free two-point correlator. We will use a regularization scheme which preserves rotational invariance and is convenient for spherical

[10] A forest is any set of non-overlapping renormalization parts.

[11] See [9], [10] for a discussion of gauge theories using the method of differential renormalization.

field theory, but one which breaks translational invariance. We regulate the short distance behavior of G by smearing the endpoints over a radius t spherical shell within a conical region $R_{M^2}(\vec{t})$, where $R_{M^2}(\vec{t})$ is the set of vectors $\vec{u}$ such that the angle between $\vec{t}$ and $\vec{u}$ is between $-\frac{1}{Mt}$ and $\frac{1}{Mt}$ (see Figure 1). The result is a regulated correlator

$$G_{M^2}(\vec{t}, \vec{t}') = \frac{1}{\int_{\hat{u}\in R_{M^2}(\vec{t})} d^3\hat{u} \int_{\hat{u}'\in R_{M^2}(\vec{t}')} d^3\hat{u}'} \int_{\substack{\hat{u}\in R_{M^2}(\vec{t}) \\ \hat{u}'\in R_{M^2}(\vec{t}')}} d^3\hat{u} d^3\hat{u}'\, G(t\hat{u}, t'\hat{u}'). \tag{17}$$

We recall that our renormalized theory is determined by the translationally invariant function $I(\vec{t} - \vec{t}_{ave}; \mu^2)$ described in the previous section. Even though our regularization scheme breaks translational invariance, the renormalized theory nevertheless remains invariant.

As the radius t increases the curvature of the angle-smearing region becomes negligible. In the limit $t \to \infty$ the region becomes a flat three-dimensional ball with radius $\frac{1}{M}$ lying in the plane perpendicular to the radial vector. In this limit our regularization is invariant under local transformations and the counterterms converge to constants independent of $\vec{t}$,

$$\lim_{t\to\infty} F^{ij}_{\nabla\phi\nabla\phi}(\vec{t}) = c^{ij,(0)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) \tag{18}$$

$$\lim_{t\to\infty} F_{\phi^2}(\vec{t}) = M^2 c^{(0)}_{\phi^2}(\tfrac{\mu^2}{M^2}) \tag{19}$$

$$\lim_{t\to\infty} F_{\phi^4}(\vec{t}) = c^{(0)}_{\phi^4}(\tfrac{\mu^2}{M^2}). \tag{20}$$

We have chosen our coefficients $c^{(0)}_A$ to be dimensionless. Although our regularization scheme is invariant under rotations about the origin, the radial vector has a special orientation which is normal to our three-dimensional ball. Our regularization scheme is therefore not isotropic. The result (as should be familiar from studies of anisotropic lattices) is that the coefficient of the kinetic term has two independent components

$$c^{ij,(0)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) = c^{\hat{t}\hat{t},(0)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) + \delta^{ij} c^{(0)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}). \tag{21}$$

Starting with the $t \to \infty$ result at lowest order, we now expand our coefficient functions in powers of $\frac{1}{Mt}$,

$$F^{ij}_{\nabla\phi\nabla\phi}(\vec{t}) = c^{ij,(0)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) + \tfrac{1}{Mt} c^{ij,(1)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) + \tfrac{1}{M^2t^2} c^{ij,(2)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) + \cdots \tag{22}$$

$$F_{\phi^2}(\vec{t}) = M^2 c^{(0)}_{\phi^2}(\tfrac{\mu^2}{M^2}) + \tfrac{M}{t} c^{(1)}_{\phi^2}(\tfrac{\mu^2}{M^2}) + \tfrac{1}{t^2} c^{(2)}_{\phi^2}(\tfrac{\mu^2}{M^2}) + \cdots \tag{23}$$

$$F_{\phi^4}(\vec{t}) = c^{(0)}_{\phi^4}(\tfrac{\mu^2}{M^2}) + \tfrac{1}{Mt} c^{(1)}_{\phi^4}(\tfrac{\mu^2}{M^2}) + \tfrac{1}{M^2t^2} c^{(2)}_{\phi^4}(\tfrac{\mu^2}{M^2}) + \cdots . \tag{24}$$

For the moment let us assume $t \geq \Lambda^{-1}$ for

$$\Lambda = m_0^z M^{1-z}, \tag{25}$$

for some fixed mass m_0 and constant z such that $0 < z < \frac{1}{2}$. In this region our dimensionless expansion parameter $\frac{1}{Mt}$ is bounded by $\frac{m_0^z}{M^z}$ and therefore diminishes uniformly as $M \to \infty$.

In general the $\frac{\mu^2}{M^2}$ dependence in the functions $c_A^{(k)}$ will contain analytic terms as $\mu^2 \to 0$ as well as logarithmically divergent terms. There are, however, no inverse powers of $\frac{\mu^2}{M^2}$. These would indicate severe infrared divergences not present in the processes we are considering, as can be deduced from the long distance behavior of the integral in (9).[12] With this we can neglect terms which vanish as $M \to \infty$,

$$F_{\phi^2}(\vec{t}) = M^2 c_{\phi^2}^{(0)}(\tfrac{\mu^2}{M^2}) + \tfrac{1}{t^2} c_{\phi^2}^{(2)}(\tfrac{\mu^2}{M^2}) \tag{26}$$

$$F^{ij}_{\nabla\phi\nabla\phi}(\vec{t}) = c^{ij,(0)}_{\nabla\phi\nabla\phi}(\tfrac{\mu^2}{M^2}) \tag{27}$$

$$F_{\phi^4}(\vec{t}) = c_{\phi^4}^{(0)}(\tfrac{\mu^2}{M^2}). \tag{28}$$

Since our regularization scheme is invariant under $M \to -M$, we have also omitted the term proportional to $c_{\phi^2}^{(1)}$ which is odd in M.

We now consider what occurs in the small region near the origin, $t \leq \Lambda^{-1}$. For the theory we are considering (and in fact for any renormalizable theory) the highest ultraviolet divergence possible is quadratic.[13] In the limit $M \to \infty$ we deduce that each F_A scales no greater than $O(M^2)$. On the other hand the volume of the region $t \leq \Lambda^{-1}$ diminishes as $O(M^{4z-4})$. Thus the total contribution from the region $t \leq \Lambda^{-1}$ scales as $O(M^{4z-2})$ and can be entirely neglected.

To summarize our results, the counterterm Lagrange density has the form

$$c^{(0)}_{\nabla\phi\nabla\phi}(\vec{\nabla}\phi(\vec{t}))^2 + c^{\hat{t}\hat{t},(0)}_{\nabla\phi\nabla\phi}(\hat{t}\cdot\vec{\nabla}\phi(\vec{t}))^2 + (M^2 c^{(0)}_{\phi^2} + \tfrac{1}{t^2} c^{(2)}_{\phi^2})\phi^2(\vec{t}) + c^{(0)}_{\phi^4}\phi^4(\vec{t}). \tag{29}$$

[12] If our theory contained bare masses m_i, similar arguments would apply for the infrared limit $\mu^2, m_i^2 \to 0$, with $\frac{m_i^2}{\mu^2}$ fixed.

[13] There may be additional logarithmic factors but this does not matter for our purposes here.

4 Spherical fields

We now examine the results of the previous section in the context of spherical field theory. We start with the spherical partial wave expansion,

$$\phi = \textstyle\sum_{l=0,1,\cdots}\sum_{n=0,\cdots l}\sum_{m=-n,\cdots n}\phi_{l,n,m}(t)Y_{l,n,m}(\theta,\psi,\varphi), \tag{30}$$

where $Y_{l,m,n}$ are four-dimensional spherical harmonics satisfying

$$\textstyle\int d^3\Omega\, Y^*_{l',n',m'}(\theta,\psi,\varphi)Y_{l,n,m}(\theta,\psi,\varphi) = \delta_{l',l}\delta_{n',n}\delta_{m',m}, \tag{31}$$

$$Y^*_{l,n,m}(\theta,\psi,\varphi) = (-1)^m Y_{l,n,-m}(\theta,\psi,\varphi). \tag{32}$$

The explicit form of $Y_{l,m,n}$ can be found in [11].[14] The integral of the free massless Lagrange density in terms of spherical fields is

$$\textstyle\int d^4\vec{t}\,\mathcal{L} = \int_0^\infty dt \left\{ \displaystyle\sum_{l,m,n} \left[(-1)^m \phi_{l,n,-m} \left[\tfrac{\partial}{\partial t}\tfrac{t^3}{2}\tfrac{\partial}{\partial t} - \tfrac{t}{2}l(l+2)\right] \phi_{l,n,m} \right] \right\}. \tag{33}$$

It can be shown that the process of angle smearing the field $\phi(\vec{t})$ is equivalent to multiplying $\phi_{l,n,m}(t)$ by an extra factor $s_l^M(t)$ where

$$s_l^M(t) = \tfrac{2Mt\left[(l+2)\sin(\frac{l}{Mt}) - l\sin(\frac{l+2}{Mt})\right]}{l(l+1)(l+2)\left[2 - Mt\sin(\frac{2}{Mt})\right]}. \tag{34}$$

For large l, $s_l^M(t)$ diminishes as l^{-2}, and so the correlator receives an extra suppression of l^{-4}. We will later use this result to estimate the contribution of high spin partial waves. The regularization of our correlator can be implemented in our Lagrange density by dividing factors of $s_l^M(t)$,

$$\begin{aligned} &\phi_{l,n,-m}\left[\tfrac{\partial}{\partial t}\tfrac{t^3}{2}\tfrac{\partial}{\partial t} - \tfrac{t}{2}l(l+2)\right]\phi_{l,n,m} \\ &\to \left[(s_l^M(t))^{-1}\phi_{l,n,-m}\right]\left[\tfrac{\partial}{\partial t}\tfrac{t^3}{2}\tfrac{\partial}{\partial t} - \tfrac{t}{2}l(l+2)\right]\left[(s_l^M(t))^{-1}\phi_{l,n,m}\right]. \end{aligned} \tag{35}$$

We now include the interaction and counterterms. We first define

$$\begin{aligned} &\left[\begin{smallmatrix} l_1,n_1,m_1;l_2,n_2,m_2 \\ l_3,n_3,m_3;l_4,n_4,m_4 \end{smallmatrix}\right] \\ &= \textstyle\int d^3\Omega\, Y_{l_1,n_1,m_1}(\theta,\psi,\varphi)Y_{l_2,n_2,m_2}(\theta,\psi,\varphi)Y_{l_3,n_3,m_3}(\theta,\psi,\varphi)Y_{l_4,n_4,m_4}(\theta,\psi,\varphi). \end{aligned} \tag{36}$$

[14] [11] deserves credit as the first discussion of radial (or covariant Euclidean) quantization, an important part of the spherical field formalism.

We can write the full functional integral as

$$\int \mathcal{D}\phi \exp\left[\int d^4\vec{t}\,\mathcal{L}\right] \propto \int \left(\prod_{l,n,m} \mathcal{D}\phi'_{l,n,m}\right) \exp\left[\int_0^\infty dt\,(L_1 + L_2 + L_3)\right], \quad (37)$$

where

$$L_1 = \sum_{l,m,n} \left[(-1)^m \left[(s_l^M(t))^{-1}\phi'_{l,n,-m}\right] \left[\frac{\partial}{\partial t}\frac{t^3}{2}\frac{\partial}{\partial t} - \frac{t}{2}l(l+2)\right] \left[(s_l^M(t))^{-1}\phi'_{l,n,m}\right]\right], \quad (38)$$

$$L_2 = \sum_{l,m,n} \left[(-1)^m \phi'_{l,n,-m} \left[\begin{array}{c} \left[-c^{(0)}_{\nabla\phi\nabla\phi} - c^{\hat{t}\hat{t},(0)}_{\nabla\phi\nabla\phi}\right] \frac{\partial}{\partial t}\frac{t^3}{2}\frac{\partial}{\partial t} \\ +c^{(0)}_{\nabla\phi\nabla\phi}\frac{t}{2}l(l+2) + t^3(M^2 c^{(0)}_{\phi^2} + \frac{1}{t^2}c^{(2)}_{\phi^2}) \end{array}\right] \phi'_{l,n,m}\right], \quad (39)$$

$$L_3 = -t^3\left(\frac{\lambda}{4!} - c^{(0)}_{\phi^4}\right) \sum_{l_i,m_i,n_i} \left[\begin{array}{c} l_1,m_1,n_1;l_2,m_2,n_2 \\ l_3,m_3,n_3;l_4,m_4,n_4 \end{array}\right] \phi'_{l_1,m_1,n_1}\phi'_{l_2,m_2,n_2}\phi'_{l_3,m_3,n_3}\phi'_{l_4,m_4,n_4}. \quad (40)$$

We have used primes in preparation for redefining the fields,

$$(s_l^M(t))^{-1}\phi'_{l,n,m} = \phi_{l,n,m}. \quad (41)$$

The Jacobian of this transformation is a constant (although infinite) and can be absorbed into the normalization of the functional integral. Now the Lagrangian L_1 has the usual free-field form in terms of $\phi_{l,n,m}$ while L_2 and L_3 are now functions of $s_l^M(t)\phi_{l,n,m}$.

With M serving as our ultraviolet regulator, the contribution of high-spin partial waves decouples for sufficiently large spin l. We can estimate the order of magnitude of this contribution in the following manner. We first identify $t^{-1}l$ (where t is the characteristic radius we are considering) as an estimate of the magnitude of the tangential momentum, p_T. For $p_T \gg M \gg t^{-1}$ our correlator scales as $\frac{M^4}{p_T^6}$. By dimensional analysis, a diagram with N_L loops and N_I internal lines will receive a contribution from partial waves with spin $\geq l$ of order

$$\left(\frac{M^4}{p_T^6}\right)^{N_I} (p_T)^{4N_L} = \left(\frac{M^4}{(t^{-1}l)^6}\right)^{N_I} \left(t^{-1}l\right)^{4N_L}. \quad (42)$$

5 One-loop examples

We will devote the remainder of our discussion to computing one-loop spherical Feynman diagrams as a check of our results. Our calculations are done both numerically and analytically. The diagrams we will consider are shown in Figures 2 and 3. We start with the two-point function in Figure 2. The amplitude can be written as $t^3 B(t)$ where

$$B(t) \propto \sum_{l,n,m} \frac{1}{t^2(l+1)} (s_l^M(t))^2. \tag{43}$$

Constants of proportionality are not important here and so we will define $B(t)$ to be equal to the right side of (43). Our results tell us that if we choose our mass counterterms appropriately, the combination

$$B(t) + M^2 c_{\phi^2}^{(0)} + \frac{1}{t^2} c_{\phi^2}^{(2)} \tag{44}$$

should be independent of t, or more succinctly,

$$B(t) + \frac{1}{t^2} c_{\phi^2}^{(2)} \tag{45}$$

is independent of t. Let us first check this analytically. In the absence of a high-spin cutoff, we can explicitly calculate the sum in (43):

$$B(t) = \frac{1}{t^2} + b(t) \tag{46}$$

where

$$b(t) = \frac{4M^2 \sin^4(\frac{1}{Mt})}{(2 - Mt \sin(\frac{2}{Mt}))^2}. \tag{47}$$

In the limit $M \to \infty$,

$$B(t) \to \frac{1}{t^2} + \frac{9}{4} M^2. \tag{48}$$

We conclude that $c_{\phi^2}^{(2)} = -1$ and $B(t) + \frac{1}{t^2} c_{\phi^2}^{(2)}$ is in fact translationally invariant.

In Figure 4 we have plotted $B(t) - \frac{1}{t^2}$, computed numerically for various values of the high-spin cutoff $J_{\max}$. We have also plotted the limiting values $b(t)$ and $\frac{9}{4}M^2$. In our plot t is measured in units of m^{-1} and $B(t) - \frac{1}{t^2}$ is in units of m^2, where m is an arbitrary mass scale such that $M = 3m$. As expected, the errors are of size $\frac{M^4 t^2}{J_{\max}^2}$. There is clearly a deviation from $\frac{9}{4}M^2$ for $t \lesssim M^{-1}$ but the integral of the deviation is negligible as $M \to \infty$.

We now turn to the four-point function in Figure 3. The amplitude can be written as $t_1^3 t_2^3 C(t_1, t_2)$ where

$$C(t_1, t_2) \propto \sum_{l,n,m} \frac{(s_l^M(t_1))^2 (s_l^M(t_2))^2}{(l+1)^2} \left[\frac{t_1^l}{t_2^{l+2}} \theta(t_2 - t_1) + \frac{t_2^l}{t_1^{l+2}} \theta(t_1 - t_2) \right]^2 . \qquad (49)$$

Again constants of proportionally are not important and so we will define $C(t_1, t_2)$ to be equal to the right side of (49). We can write $C(t_1, t_2)$ in terms of the regulated correlator $G_{M^2}(\vec{t}_1, \vec{t}_2)$, [15]

$$C(t_1, t_2) \propto \int d^3\hat{t}_1 d^3\hat{t}_2 \left[G_{M^2}(\vec{t}_1, \vec{t}_2) \right]^2 \propto \int d^3\hat{t}_1 \left[G_{M^2}(\vec{t}_1, \vec{t}_2) \right]^2 . \qquad (50)$$

Since the coupling constant counterterm

$$c^{(0)}_{\phi^4} \delta^4(\vec{t}_1 - \vec{t}_2) \qquad (51)$$

is translationally invariant, the amplitude by itself should be translationally invariant. Let us define

$$\int d^4\vec{t}_2 e^{-i\vec{p}\cdot(\vec{t}_1 - \vec{t}_2)} \left[G_{M^2}(\vec{t}_1, \vec{t}_2) \right]^2 = f(\vec{p}^2), \qquad (52)$$

so that

$$\int d^4\vec{t}_2 e^{i\vec{p}\cdot\vec{t}_2} \left[G_{M^2}(\vec{t}_1, \vec{t}_2) \right]^2 = e^{i\vec{p}\cdot\vec{t}_1} f(\vec{p}^2). \qquad (53)$$

Integrating over $\hat{t}_1$, we find

$$\int dt_2\, t_2^2 J_1(pt_2) C(t_1, t_2) \propto \tfrac{1}{t_1} J_1(pt_1) f(\vec{p}^2). \qquad (54)$$

Let us define

$$C(t) = \int dt_2\, t_2^2 J_1(pt_2) C(t, t_2). \qquad (55)$$

We now check that in fact

$$C(t) \propto \tfrac{1}{t_1} J_1(pt_1). \qquad (56)$$

[15]We recall that the regulated correlator goes with $\phi'_{l,n,m}$ rather than $\phi_{l,n,m}$. But this is not important here since $\phi'_{0,0,0} = \phi_{0,0,0}$.

In the absence of a high-spin cutoff, we find that $C(t)$ is given by[16]

$$C(t) = \tfrac{1}{t_1} J_1(pt_1) \left[\tfrac{1}{2}\log\tfrac{M^2}{p^2} + c\right] + \cdots , \tag{57}$$

where the ellipsis represents terms which vanish as $M \to \infty$ and

$$c = 324 \left[\int_0^{1/2} dk \left(\tfrac{(\sin k - k\cos k)^4}{4k^{13}} - \tfrac{1}{324k}\right) + \int_{1/2}^{\infty} dk \tfrac{(\sin k - k\cos k)^4}{4k^{13}}\right]. \tag{58}$$

In Figure 5 we plot $C(t)$ for different values of the high-spin cutoff $J_{\max}$ as well as the large-M limit value

$$C_1(t) = \tfrac{1}{t_1} J_1(pt_1) \left[\tfrac{1}{2}\log\tfrac{M^2}{p^2} + c\right]. \tag{59}$$

In our plot t is measured in units of p^{-1} and $M = 3p$. From (42) the expected error is of size $\frac{M^8 t^8}{J_{\max}^8}$. We see that the data is consistent with the results expected. Again the deviation for $t \lesssim M^{-1}$ integrates to a negligible contribution as $M \to \infty$.

6 Summary

We have examined several important features of non-perturbative renormalization in the spherical field formalism and answered the three questions posed in the introduction. Ultraviolet divergences can be cancelled by a finite number of local counterterms in a manner such that the renormalized theory is translationally invariant. Using angle-smearing regularization we find that the counterterms for ϕ^4 theory in four dimensions can be parameterized by five unknown constants as shown in (29). Aside from our remarks about Ward identity constraints in gauge theories, the extension to other field theories is straightforward. We hope that these results will be useful for future studies of general renormalizable theories by spherical field techniques.

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[16]This calculation is somewhat lengthy. Details can be obtained upon request from the authors.

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Figures

Figure 1. Sketch of the angle-smearing region (three-dimensional rendering).
Figure 2. One-loop two-point correlator for $\phi_{0,0,0}$.
Figure 3. One-loop four-point correlator for $\phi_{0,0,0}$.
Figure 4. Plot of $B(t) - \frac{1}{t^2}$.
Figure 5. Plot of $C(t)$.

Figure 1

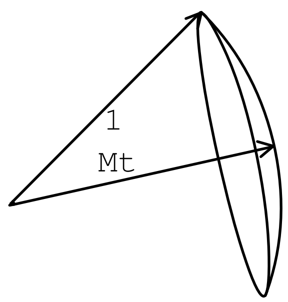

Figure 2

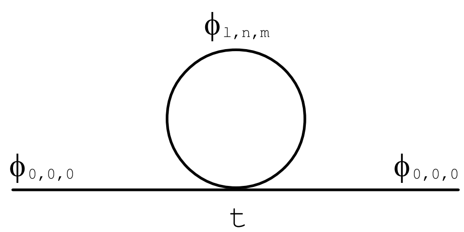

Figure 3

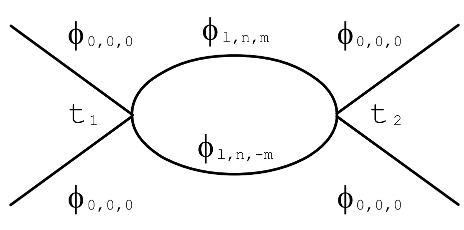

Figure 4

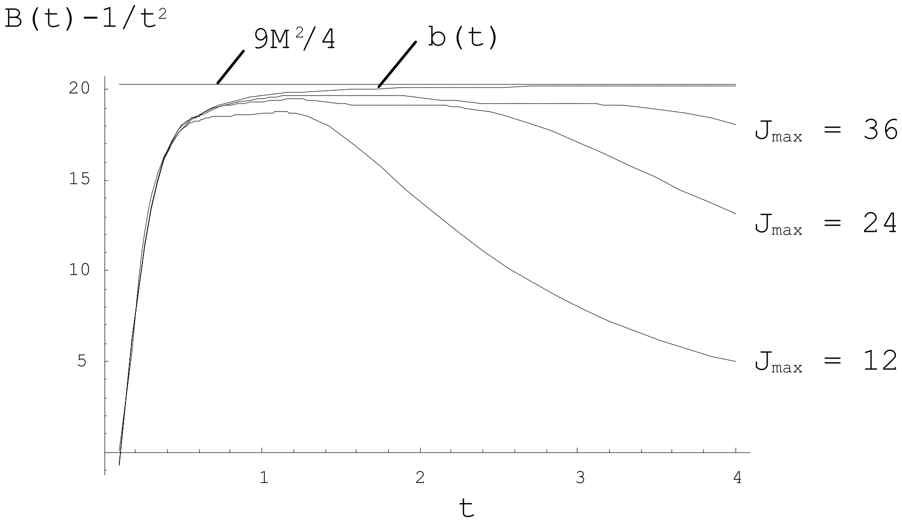

Figure 5